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Covariance matrices of self-affine measures

Abstract

In this paper we derive a formula for a covariance matrix of any self-affine measure, i.e. a probability measure μ satisfying

$$\mu = \sum_{k=1}^{l} p_k \mu \circ S_k^{-1} ,$$

where $\{S_k(\mathbf{x}) = A_k\mathbf{x} + \mathbf{b}_k\}_{1\leq k\leq l}$ is a family of affine contractive maps and $\{p_k\}_{1\leq k\leq l}$ is a set of probability weights. In particular if for every k, $A_k = A$ then the formula will be have the following form

$$D^2X = [I \otimes I - A \otimes A]^{-1}\mathcal{D}^2\mathcal{B},$$

where D^2X denote the covariance matrix of the measure μ and $\mathcal{D}^2\mathcal{B}$ denote a covariance matrix of a discret random variable $\mathcal{B}$ with values $\mathbf{b}_k$, and corresponding probabilites p_k.

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1 Introduction

We will consider a invariant probability measure on $\mathbb{R}^d$

$$\mu = \sum_{k=1}^{l} p_k \mu \circ S_k^{-1} , \tag{1}$$

where $\{S_k\}_{1\leq k\leq l}$ is a family of contractive maps and $\{p_k\}_{1\leq k\leq l}$ is a set of probability weights. It is often assumed that the maps are similitudes. We will make a more general assumption that the maps are affine contractions on $\mathbb{R}^d$; i.e. $S_k(\mathbf{x}) = A_k\mathbf{x} + \mathbf{b}_k$ and the operator norm $\|A_k\| < 1$ for all k.

The definition (1) was introduced by Hutchinson [H]. But an example of such measures has been studied for a long time in the context of Bernoulli convolution, i.e. the example of an invariant measure on the real line

$$\mu = \frac{1}{2}(\mu \circ S_1^{-1} + \mu \circ S_2^{-1}), \tag{2}$$

where $S_1(x) = \beta(x+1)$, $S_2(x) = \beta(x-1)$ for $\beta \in (0,1)$. It remains difficult open problem to characterize the set of β for which μ is absolutely continuous. An another example has been studied in great detail in wavelet theory in connection with the dilatation equation

$$f(x) = \sum_{k=1}^{l} c_k f(2x - (k-1)). \tag{3}$$

The function f can be considered as the density function of the corresponding absolutely continuous self-affine measure μ for $S_k(x) = \frac{1}{2}(x + (k-1))$ and $p_k = \frac{1}{2}c_k$. In wavelet theory the c_k may be negative but $\sum c_k$ must be 2. The invariant measures have a natural connection with fractal geometry [F], because their supports are compact invariant sets

$$K = \bigcup_{k=1}^{l} S_k(K). \tag{4}$$

These measures arise also in another areas of mathematics.

A fundamental method for studing these measures is the Fourier transform [S]. Our goal is to show that we can use another probabilistic tools, not only characteristic function, to investigate the self-affine measures. We derive a formula for covariance matrices and give an example of an investigation of measures on Sierpinski triangle.

2 Covariance matrices of self-affine measures

The invariant measure μ satisfies the following identity

$$\int_{\mathbb{R}^d} f d\mu = \sum_{k=1}^{l} p_k \int_{\mathbb{R}^d} f \circ S_k d\mu, \tag{5}$$

where f is any continuous function on $\mathbb{R}^d$ [B]. We will denote by X some d-dimensional random variable with respect to the probability distribution μ. We apply the identity (5) to the coordinate functions $e_i^\star(\mathbf{x}) = x_i$ of the point $\mathbf{x} = (x_i)_{1 \le i \le d} \in \mathbb{R}^d$. By the above a vector of expected values EX will be equal

$$
\begin{aligned}
EX &= (\int_{\mathbb{R}^d} e_i^\star(\mathbf{x}) d\mu)_{1\le i\le d} = (\sum_{k=1}^{l} p_k \int_{\mathbb{R}^d} e_i^\star(A_k\mathbf{x}+\mathbf{b}_k) d\mu)_{1\le i\le d} \\
&= \sum_{k=1}^{l} p_k E(A_k X + \mathbf{b}_k) = \sum_{k=1}^{l} p_k A_k EX + \sum_{k=1}^{l} p_k \mathbf{b}_k \qquad (6)
\end{aligned}
$$

The above relation gives

$$
[I - \sum_{k=1}^{l} p_k A_k] EX = \sum_{k=1}^{l} p_k \mathbf{b}_k, \qquad (7)
$$

where I is identity matrix on $\mathbb{R}^d$. The sum $\sum_{k=1}^{l} p_k \mathbf{b}_k$ is a vector of expected values of a d-dimensional random variable $\mathcal{B}$ with values $\mathbf{b}_k$, and corresponding probabilities p_k. Since $\|A_k\| < 1$ for all k then $\|\sum_{k=1}^{l} p_k A_k\| < 1$. It follows that 1 is not an eigenvalue of the operator $\sum_{k=1}^{l} p_k A_k$. For this reason the operator $I - \sum_{k=1}^{l} p_k A_k$ will be invertible. We can rewrite (7) as

$$
EX = [I - \sum_{k=1}^{l} p_k A_k]^{-1} \mathcal{E}\mathcal{B}. \qquad (8)
$$

This means that the expected value of X linearly depend on the expected value of $\mathcal{B}$.

Let $X \otimes X = [x_i x_j]_{1\le i,j\le d}$ denote the second order tensor build from the coordinates. Using (5) a matrix of second order moments $E(X \otimes X)$ will be equal

$$
\begin{aligned}
E(X \otimes X) &= [\int_{\mathbb{R}^d} e_i^\star(\mathbf{x}) e_j^\star(\mathbf{x}) d\mu]_{1\le i,j\le d} \\
&= \sum_{k=1}^{l} p_k [\int_{\mathbb{R}^d} e_i^\star(A_k\mathbf{x}+\mathbf{b}_k) e_j^\star(A_k\mathbf{x}+\mathbf{b}_k) d\mu]_{1\le i,j\le d} \\
&= \sum_{k=1}^{l} p_k E((A_k X + \mathbf{b}_k) \otimes (A_k X + \mathbf{b}_k)) \\
(9) &= \sum_{k=1}^{l} p_k [(A_k \otimes A_k) E(X \otimes X) + \mathbf{b}_k \otimes A_k EX + A_k EX \otimes \mathbf{b}_k + \mathbf{b}_k \otimes \mathbf{b}_k]
\end{aligned}
$$

Therefore

$$[I\otimes I-\sum_{k=1}^{l}p_k(A_k\otimes A_k)]E(X\otimes X)=\sum_{k=1}^{l}p_k(\mathbf{b}_k\otimes A_kEX+A_kEX\otimes\mathbf{b}_k+\mathbf{b}_k\otimes\mathbf{b}_k) \tag{10}$$

The operator norm of $A_k \otimes A_k$ is less than 1 on $\mathbb{R}^d \otimes \mathbb{R}^d$, so, by the same argument as early, we get that the operator $I \otimes I - \sum_{k=1}^{l} p_k(A_k \otimes A_k)$ is invertible and we obtain

$$\begin{aligned} E(X\otimes X) &= [I\otimes I-\sum_{k=1}^{l}p_k(A_k\otimes A_k)]^{-1}\times \\ & \sum_{k=1}^{l}p_k(\mathbf{b}_k\otimes A_kEX+A_kEX\otimes\mathbf{b}_k+\mathbf{b}_k\otimes\mathbf{b}_k). \end{aligned} \tag{11}$$

Substituting (8) into $D^2X = E(X \otimes X) - EX \otimes EX$ we can obtain a formula for the covariance matrix of X. But in the general case it will be complicated. This formula takes a surprising simple form when we assumed that all affine maps have the same linear part, i.e. all $A_k = A$. Under this assumption, using the standard tensor calculus we get

$$\begin{aligned} D^2X &= [I\otimes I-A\otimes A]^{-1}\{[I\otimes A(I-A)^{-1}](\mathcal{E}\mathcal{B}\otimes\mathcal{E}\mathcal{B}) \\ &\quad +[A(I-A)^{-1}\otimes I](\mathcal{E}\mathcal{B}\otimes\mathcal{E}\mathcal{B})+\mathcal{E}(\mathcal{B}\otimes\mathcal{B})\} \\ &\quad -[(I-A)^{-1}\otimes(I-A)^{-1}](\mathcal{E}\mathcal{B}\otimes\mathcal{E}\mathcal{B}) \\ &= [I\otimes I-A\otimes A]^{-1}(\mathcal{E}(\mathcal{B}\otimes\mathcal{B})-\mathcal{E}\mathcal{B}\otimes\mathcal{E}\mathcal{B}) \end{aligned} \tag{12}$$

Notice now that the expression $\mathcal{E}(\mathcal{B}\otimes\mathcal{B})-\mathcal{E}\mathcal{B}\otimes\mathcal{E}\mathcal{B}$ is the covariance matrix of $\mathcal{B}$. Thus we obtain

$$D^2X = [I\otimes I - A\otimes A]^{-1}\mathcal{D}^2\mathcal{B}. \tag{13}$$

In other words we obtained the following proposition.

PROPOSITION

Assume that μ is a self-affine measure on $\mathbb{R}^d$ for a family of linear contractions $S_k(\mathbf{x}) = A\mathbf{x} + \mathbf{b}_k$, $1 \le k \le l$. Let X be some random variable with respect to the probability ditribution μ. Then the covariance matrix of the random variable X

$$D^2X = [I\otimes I - A\otimes A]^{-1}\mathcal{D}^2\mathcal{B}, \tag{14}$$

where $\mathcal{D}^2\mathcal{B}$ denote the covariance matrix of the random variable $\mathcal{B}$. $\square$

Remark. When all affine maps S_k have the same linear part then not only the expected value of X linearly depend on expected value of $\mathcal{B}$ but also covariance matrix of X linearly depend on the covariance matrix of $\mathcal{B}$.

If the matrix A is diagonal then diagonal is the matrix $[I \otimes I - A \otimes A]^{-1}$. Therefore we get the simple corollary.

COROLLARY

If under the assumptions of Proposition we assume additionaly that the matrix A is diagonal then $X_i = e_i^\star(X)$ and $X_j = e_j^\star(X)$ are uncorrelated if and only if uncorrelated are $(e_i^\star(\mathbf{b}_k))_{1 \le k \le l}$ and $(e_j^\star(\mathbf{b}_k))_{1 \le k \le l}$. $\square$

In other words we have obtained the following law

$$\int_{\mathbb{R}^d} x_i x_j d\mu = \int_{\mathbb{R}^d} x_i d\mu \int_{\mathbb{R}^d} x_j d\mu \quad iff \quad \sum_{k=1}^{l} p_k e_i^\star(\mathbf{b}_k) e_j^\star(\mathbf{b}_k) = \sum_{k=1}^{l} p_k e_i^\star(\mathbf{b}_k) \sum_{k=1}^{l} p_k e_j^\star(\mathbf{b}_k).$$

3 Example

Consider Sierpinski triangle with vertices at the points $(0,0)$, $(1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. The Sierpinski triangle is an invariant compact set of three cotractions on $\mathbb{R}^2$: $S_1(\mathbf{x}) = \frac{1}{2}\mathbf{x}$, $S_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + (\frac{1}{2}, 0)$ and $S_3(\mathbf{x}) = \frac{1}{2}\mathbf{x} + (\frac{1}{4}, \frac{\sqrt{3}}{4})$. In this case the matrix $A = \frac{1}{2}I$. Let μ denote an invariant measure for weigths p_1, p_2 and p_3. The expected value $\mathcal{E}\mathcal{B} = (\frac{1}{2}p_2 + \frac{1}{4}p_3, \frac{\sqrt{3}}{4}p_3)$. The matrix $[I - A]^{-1} = 2I$. By the (8) we get $\int_{\mathbb{R}^2} x_1 d\mu = p_2 + \frac{1}{2}p_3$ and $\int_{\mathbb{R}^2} x_2 d\mu = \frac{\sqrt{3}}{2}p_3$. By the (14) we can obtain terms of the matrix $\mathcal{D}^2\mathcal{B}$ and D^2X. In particular

$$(15) \quad \sum_{k=1}^{3} p_k e_1^\star(\mathbf{b}_k) e_2^\star(\mathbf{b}_k) - \sum_{k=1}^{3} p_k e_1^\star(\mathbf{b}_k) \sum_{k=1}^{3} p_k e_2^\star(\mathbf{b}_k) = \frac{\sqrt{3}}{16} p_3 (p_1 - p_2).$$

By corollary, if $p_1 = p_2$ then the random variables X_1, X_2 are uncorrelated and

$$\begin{aligned} \int_{\mathbb{R}^2} x_1 x_2 d\mu &= \int_{\mathbb{R}^2} x_1 d\mu \int_{\mathbb{R}^2} x_2 d\mu \\ (16) \qquad &= (p_2 + \frac{1}{2}p_3)\frac{\sqrt{3}}{2}p_3 = \frac{\sqrt{3}}{4}p_3. \end{aligned}$$

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